$$\begin{array}{lllll}
(n = 4) & B^i_{jklm}; & B^i_{jklmw_1w_2}; & B^i_{jklmw_1\ldots w_4}; & \ldots \\
(n = 6) & B^i_{jkl}; & B^i_{jklw_1w_2}; & B^i_{jklw_1\ldots w_4}; & \ldots \\
(n = 8) & B^i_{jkl}; & B^i_{jklm}; & B^i_{jklmw_1w_2}; & \ldots \\
 & \vdots & \vdots & \vdots &
\end{array}$$

Each of these sequences begins with the complete conformal curvature tensor; and those tensors whose components appear along the main diagonal of the above infinite matrix of components, are each of weight $-\frac{2}{n+2}$. It is evident that the tensors whose components appear in the above sequences, when combined with the fundamental conformal tensor G, constitute a complete set of invariants of the conformal Riemann space.

INTRANSITIVE GROUPS OF MOTIONS

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The continuous groups of motions of a Riemannian manifold have been the subject of study of geometers for many years, and more recently their rôle in theoretical physics has been revealed. As a basis for the classification of such groups Fubini[1] announced the theorem:

If a G_r is an intransitive group of motions of a V_n and the minimum invariant varieties are of dimensionality q, a coördinate system can be found in which G_r is a transitive group on q variables. The proof of this theorem given by Fubini is satisfactory for the case when $q = n - 1$, but it is not convincing for $q < n - 1$. It is the purpose of this note to give a proof of this theorem for spaces with a definite quadratic form (the case considered by Fubini) and to give sufficient conditions for it in the case when the quadratic form is indefinite. Throughout the paper the variables are understood to be real.

1. *Equations of Continuous Groups.*—We consider a continuous group of order r, say G_r, expressed in terms of n coördinates $x^1, \ldots, x^n$, and denote by

$$X_a f = \xi^\alpha_a \frac{\partial f}{\partial x^\alpha} \qquad \begin{pmatrix} a = 1, \ldots, r; \\ \alpha = 1, \ldots, n \end{pmatrix} \tag{1.1}$$

the symbols of the group. Here, and in what follows, a repeated index in a term (in this case α) indicates that the term is the sum of terms as the index takes on all its values. The fundamental theorem of continuous groups is that a necessary and sufficient condition that X_af are the generators of a group is that

$$(X_a, X_b)f = c_{ab}{}^e X_e f \qquad (a, b, e = 1, \ldots, r) \tag{1.2}$$

where the c's are constants satisfying the conditions

$$c_{ab}{}^e = -c_{ba}{}^e, \tag{1.3}$$

$$c_{ab}{}^e c_{ed}{}^f + c_{bd}{}^e c_{ea}{}^f + c_{da}{}^e c_{eb}{}^f = 0. \tag{1.4}$$

Substituting from (1.1) in (1.2), we obtain

$$\xi_a^\alpha \frac{\partial \xi_b^\beta}{\partial x^\alpha} - \xi_b^\alpha \frac{\partial \xi_a^\beta}{\partial x^\alpha} = c_{ab}{}^e \xi_e^\beta. \tag{1.5}$$

We denote by q the generic rank of the matrix

$$M = \| \xi_a^\alpha \|, \tag{1.6}$$

that is, the rank for general values of the x's. Evidently $q \lesseqgtr r$. If $q = n$, the group is transitive; if $q < n$, intransitive. If $q < r$, we may without loss of generality assume that the matrix $\| \xi_h^\alpha \|$, for $h = 1, \ldots, q$, is of rank q, and put

$$\xi_p^\alpha = \varphi_p^h \xi_h^\alpha \qquad \begin{pmatrix} h = 1, \ldots, q; \\ p = q + 1, \ldots, r \end{pmatrix} \tag{1.7}$$

where the quantities φ_p^h are functions of the x's. Then (1.5) may be written

$$\xi_a^\alpha \frac{\partial \xi_b^\beta}{\partial x^\alpha} - \xi_b^\alpha \frac{\partial \xi_a^\beta}{\partial x^\alpha} = (c_{ab}{}^h + c_{ab}^p \varphi_p^h) \xi_h^\beta. \tag{1.8}$$

If we take $b = p$ in (1.8) and replace ξ_p^β by its expression (1.7), the result is equivalent, in consequence of equations of the form (1.8) in a and h, to

$$X_a \varphi_p^h = \Phi_{ap}^h, \tag{1.9}$$

where

$$\Phi_{ap}^h = c_{ap}{}^h + c_{ap}{}^s \varphi_s^h - \varphi_p^i (c_{ai}{}^h + c_{ai}{}^s \varphi_s^h) \begin{pmatrix} a = 1, \ldots, r; \\ h, i = 1, \ldots, q; \\ p, s = q + 1, \ldots, r \end{pmatrix} \tag{1.10}$$

When $q < n$, that is when G_r is intransitive, the equations $X_af = 0$ form a completely integrable system, and because of (1.7) there are q

independent equations in the set, and consequently there are $n - q$ independent solutions $\varphi^\sigma(x)$. Without loss of generality, we may assume that the determinant $\left|\frac{\partial\varphi^\sigma}{\partial x^\tau}\right|$ for $\tau = q + 1, \ldots, n$ is not zero. If we effect the transformation

$$x'^\lambda = x^\lambda,\ x'^\sigma = \varphi^\sigma(x) \begin{pmatrix} \lambda = 1, \ldots, q; \\ \sigma = q + 1, \ldots, n \end{pmatrix}, \tag{1.11}$$

and note that the functions ξ_a^α transform as the components of a contravariant vector, that is,

$$\xi'^\alpha_a = \xi_a^\beta \frac{\partial x'^\alpha}{\partial x^\beta}, \tag{1.12}$$

then in the new coördinate system, which we now call x^α (dropping primes), we have

$$\xi_a^\sigma = 0 \begin{pmatrix} a = 1, \ldots, r; \\ \sigma = q + 1, \ldots, n \end{pmatrix}. \tag{1.13}$$

Since in the new coördinate system the matrix $\|\xi_h^\lambda\|$ for $h = 1, \ldots, q$ is of rank q, a set of functions $\xi_{h\lambda}$ are uniquely defined by

$$\xi_{h\lambda}\xi_h^\mu = \delta_\lambda^\mu,\ \xi_{h\lambda}\xi_i^\lambda = \delta_{ih} \qquad (\lambda, \mu, i, h = 1, \ldots, q), \tag{1.14}$$

where

$$\delta_\lambda^\mu = 1 \text{ or } 0, \text{ as } \mu = \lambda \text{ or } \mu \neq \lambda$$
$$\delta_{ih} = 1 \text{ or } 0, \text{ as } i = h \text{ or } i \neq h.$$

In this coördinate system equations (1.8), when a and b take values 1 to q reduce to

$$\xi_h^\lambda \frac{\partial \xi_i^\mu}{\partial x^\lambda} - \xi_i^\lambda \frac{\partial \xi_h^\mu}{\partial x^\lambda} = (c_{hi}^j + c_{hi}^p \varphi_p^j)\xi_j^\mu \qquad \begin{pmatrix} (\lambda, \mu, h, i, j = 1, \ldots, q; \\ p = q + 1, \ldots, r \end{pmatrix}, \tag{1.15}$$

and equations (1.9) for $a = 1, \ldots, q$ may be written in the form

$$\frac{\partial\varphi_p^h}{\partial x^\lambda} = \Phi_{ip}^h\, \xi_{i\lambda}, \tag{1.16}$$

in consequence of (1.14).

We define functions $L_{\beta\mu}^\alpha$ by

$$L_{\beta\mu}^\lambda = \xi_h^\lambda \frac{\partial \xi_{h\mu}}{\partial x^\beta} = -\xi_{h\mu}\frac{\partial \xi_h^\lambda}{\partial x^\beta},\ L_{\beta\mu}^\sigma = 0 \begin{pmatrix} \lambda, \mu = 1, \ldots, q; \\ \beta = 1, \ldots, n; \\ \sigma = q + 1, \ldots, n \end{pmatrix}, \tag{1.17}$$

(The reason for this definition will appear in §2.) From (1.17) we have, in consequence of (1.14)

$$\frac{\partial \xi_h^\lambda}{\partial x^\beta} + \xi_h^\mu L_{\beta\mu}^\lambda = 0 \tag{1.18}$$

and

$$\frac{\partial \xi_{h\lambda}}{\partial x^\beta} - \xi_{h\mu} L_{\beta\lambda}^\mu = 0. \tag{1.19}$$

From (1.17) and (1.15) we have

$$L_{\mu\nu}^\lambda - L_{\nu\mu}^\lambda = (c_{hi}^j + c_{hi}^p \varphi_p^j)\xi_{h\nu}\xi_{i\mu}\xi_j^\lambda. \tag{1.20}$$

Since the determinant $|\xi_h^\lambda|$ is different from zero, the conditions of integrability of (1.18) are

$$\frac{\partial L_{\alpha\mu}^\lambda}{\partial x_\beta} - \frac{\delta L_{\beta\mu}^\lambda}{\partial x^\alpha} + L_{\alpha\mu}^\pi L_{\beta\pi}^\lambda - L_{\beta\mu}^\pi L_{\alpha\pi}^\lambda = 0 \quad \begin{pmatrix} \alpha, \beta = 1, \ldots, n; \\ \lambda, \mu, \pi = 1, \ldots, q \end{pmatrix}. \tag{1.21}$$

These are necessarily identities, as may be verified by substitution from (1.17).

If we replace β in (1.21) by ν for $\nu = 1, \ldots, q$, and subtract from these equations the corresponding ones obtained by interchanging μ and ν, we obtain

$$\frac{\partial L_{\alpha\mu}^\lambda}{\partial x^\nu} - \frac{\partial L_{\alpha\nu}^\lambda}{\partial x^\mu} = \frac{\partial}{\partial x^\alpha}(L_{\nu\mu}^\lambda - L_{\mu\nu}^\lambda) + L_{\alpha\pi}^\lambda(L_{\nu\mu}^\pi - L_{\mu\nu}^\pi) - L_{\alpha\mu}^\pi L_{\nu\pi}^\lambda + L_{\alpha\nu}^\pi L_{\mu\pi}^\lambda.$$

If we put

$$L_{\alpha\mu\nu}^\lambda = \frac{\partial L_{\alpha\nu}^\lambda}{\partial x^\mu} - \frac{\partial L_{\alpha\mu}^\lambda}{\partial x^\nu} + L_{\alpha\nu}^\pi L_{\pi\mu}^\lambda - L_{\alpha\mu}^\pi L_{\pi\nu}^\lambda, \tag{1.22}$$

the preceding equations may be written

$$L_{\alpha\mu\nu}^\lambda = \frac{\partial}{\partial x^\alpha}(L_{\mu\nu}^\lambda - L_{\nu\mu}^\lambda) + L_{\alpha\pi}^\lambda(L_{\mu\nu}^\pi - L_{\nu\mu}^\pi) + L_{\alpha\nu}^\pi(L_{\pi\mu}^\lambda - L_{\mu\pi}^\lambda) - L_{\alpha\mu}^\pi(L_{\pi\nu}^\lambda - L_{\nu\pi}^\lambda).$$

In consequence of (1.20), (1.18) and (1.19) these equations are reducible to

$$L_{\alpha\mu\nu}^\lambda = c_{hi}^p \frac{\partial \varphi_p^j}{\partial x^\alpha}\xi_{h\nu}\xi_{i\mu}\xi_j^\lambda. \tag{1.23}$$

We inquire whether there exists a transformation of coördinates of the form

$$x^\lambda = \varphi^\lambda(x'^1, \ldots, x'^n),\ x^\sigma = x'^\sigma \quad \begin{pmatrix} \lambda = 1, \ldots, q; \\ \sigma = q+1, \ldots, n \end{pmatrix} \tag{1.24}$$

such that in the new coördinate system ξ'^{λ}_h are independent of x'^{σ} for $\sigma = q + 1, \ldots, n$. The inverse of (1.24) is of the form

$$x'^{\lambda} = \bar{\varphi}^{\lambda}(x^1, \ldots, x^n),\ x'^{\sigma} = x^{\sigma}. \tag{1.25}$$

Then because of (1.13) we have $\xi'^{\sigma}_h = 0$. If the desired coördinate system exists, on differentiating with respect to x'^{σ} the first set of the equations

$$\xi^{\mu}_h = \xi'^{\lambda}_h \frac{\partial x^{\mu}}{\partial x'^{\lambda}},\ \xi'^{\lambda}_h = \xi^{\mu}_h \frac{\partial x'^{\lambda}}{\partial x^{\mu}}, \tag{1.26}$$

and making use of (1.18), the second set of (1.26) and the fact that the determinant $|\xi^{\lambda}_h|$ is not zero, we obtain the following equations of condition:

$$\frac{\partial x'^{\nu}}{\partial x^{\lambda}} \frac{\partial^2 x^{\mu}}{\partial x'^{\nu} \partial x'^{\sigma}} + L^{\mu}_{\alpha\lambda} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} = 0. \tag{1.27}$$

For the transformation (1.24) we have

$$\frac{\partial x'^{\nu}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial x'^{\mu}} = \delta^{\nu}_{\mu}.$$

Hence if we multiply (1.27) by $\frac{\partial x^{\lambda}}{\partial x'^{\pi}}$ and sum for λ, we obtain, with change of indices,

$$\frac{\partial^2 x^{\mu}}{\partial x'^{\nu} \partial x'^{\sigma}} + L^{\mu}_{\alpha\lambda} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x^{\lambda}}{\partial x'^{\nu}} = 0. \tag{1.28}$$

When we differentiate this equation with respect to x'^{π} for $\pi = 1, \ldots, q$, reduce the resulting equation by means of (1.28) and subtract from the result the equation obtained by interchanging ν and π, we obtain

$$\frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x^{\lambda}}{\partial x'^{\nu}} \frac{\partial x^{\theta}}{\partial x'^{\pi}} L^{\mu}_{\alpha\lambda\theta} = 0. \tag{1.29}$$

When we differentiate equation (1.28) with respect to x'^{τ} for $\tau = q + 1, \ldots, n$, reduce the resulting equation by means of (1.28) and subtract from the result the equation obtained by interchanging σ and τ, we obtain an equation which vanishes identically because of (1.21).

2. *Groups of Motions.*—If the variables x^{α} are interpreted as the coördinates of points of an n dimensional space, V_n, the space is Riemannian when a metric is assigned by a quadratic differential form $g_{\alpha\beta}dx^{\alpha}dx^{\beta}$ where without loss of generality it may be assumed that $g_{\alpha\beta} = g_{\beta\alpha}$, and where the determinant $|g_{\alpha\beta}|$ is not zero. In order that the space admit a given G_r as a group of motions, that is for each infinitesimal transformation of the group $\delta(g_{\alpha\beta}dx^{\alpha}dx^{\beta}) = 0$, it is necessary and sufficient that the following equations of Killing hold in each coördinate system:[2]

$$\xi_a^\alpha \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + g_{\beta\alpha}\frac{\partial \xi_a^\alpha}{\partial x^\gamma} + g_{\gamma\alpha}\frac{\partial \xi_a^\alpha}{\partial x^\beta} = 0 \begin{pmatrix} \alpha, \beta, \gamma = 1, \ldots, n; \\ a = 1, \ldots, r \end{pmatrix}. \tag{2.1}$$

If $q < n$, then in the special coördinate system for which (1.13) hold, these equations become for $a = 1, \ldots, q$

$$\xi_h^\lambda \frac{\partial g_{\beta\gamma}}{\partial x^\lambda} + g_{\beta\lambda}\frac{\partial \xi_h^\lambda}{\partial x_\gamma} + g_{\gamma\lambda}\frac{\partial \xi_h^\lambda}{\partial x^\beta} = 0. \quad \begin{pmatrix} \lambda, h = 1, \ldots, q; \\ \beta, \gamma = 1, \ldots, n \end{pmatrix} \tag{2.2}$$

If these equations are multiplied by $\xi_{h\mu}$ and h is summed, we have, in consequence of (1.14) and (1.18), the equivalent set of equations

$$\frac{\partial g_{\beta\gamma}}{\partial x^\mu} = g_{\beta\lambda}L^\lambda_{\gamma\mu} + g_{\gamma\lambda}L^\lambda_{\beta\mu}. \tag{2.3}$$

When $q = r$, equations (2.2) are the only conditions. However, when $q < r$, we have in addition (2.1) when a takes the values $q + 1$ to r. When in these equations we substitute from (1.7) and (1.13), we obtain with the aid of (2.3) the following equivalent system:

$$\xi_h^\lambda \left(g_{\beta\lambda}\frac{\partial \varphi_p^h}{\partial x_\gamma} + g_{\lambda\gamma}\frac{\partial \varphi_p^h}{\partial x^\beta}\right) = 0. \begin{pmatrix} \lambda, h = 1, \ldots, q; \\ p = q + 1, \ldots, r; \\ \beta, \gamma = 1, \ldots, n \end{pmatrix} \tag{2.4}$$

Thus when $q < r$, the systems (2.2) and (2.4) are the conditions of the problem.

Expressing the condition of integrability of equations (2.3), we obtain

$$g_{\beta\lambda}L^\lambda_{\gamma\mu\nu} + g_{\gamma\lambda}L^\lambda_{\beta\mu\nu} = 0,$$

where $L^\lambda_{\gamma\mu\nu}$ is defined by (1.22). When $q = r$ and $r \lesseqgtr n$, it follows from (1.23) that $L^\lambda_{\beta\mu\nu} = 0$. Hence, equations (2.3), which are the only conditions to be satisfied in this case, form a completely integrable system. Hence a solution is determined by arbitrary initial values of the g's, which when $n > r$ may be arbitrary functions of the variables $x^{r+1}, \ldots, x^n$. Hence we have the theorem:[3]

Any r-parameter continuous group on n variables such that the generic rank of the matrix $\| \xi_a^\alpha \|$ *is* $r(\lesseqgtr n)$ *is the group of motions of a Riemannian manifold whose fundamental tensor* $g_{\alpha\beta}$ *involves* $n(n + 1)/2$ *arbitrary functions of* $n - r$ *variables.*

When $n = r$, that is when G_r is simply transitive, there are $n(n + 1)/2$ arbitrary constants; the theorem for this particular case was established by Bianchi.

It is known that when a group is Abelian, that is, when all of the constants of composition are zero, the coördinates can be chosen so that

$$\xi_h^\lambda = \delta_h^\lambda,\ \xi_h^\sigma = 0 \begin{pmatrix}\lambda, h = 1, \ldots, q \\ \sigma = q+1, \ldots, n\end{pmatrix}.$$

In this case the functions Φ_{ap}^h (1.10) are zero and from (1.16) it follows that the quantities φ_p^h are independent of $x^1, \ldots, x^q$. Then from (2.4) we have $g_{\lambda\mu}\dfrac{\partial \varphi_p^\mu}{\partial x^\sigma} = 0$, for $\sigma = q+1, \ldots, n$. Hence the φ's are constants unless the determinant is zero, which is impossible if the fundamental quadratic form is definite.[4] Consequently only when $q = r$ is an intransitive Abelian group a group of motions. Moreover, every such Abelian group is a group of motions, and the quantities $g_{\alpha\beta}$ are arbitrary functions of $x^{q+1}, \ldots, x^n$, as follows from (2.3), since the quantities $L_{\alpha\mu}^\lambda$ are zero in this case. When the group is Abelian and simply transitive, the $g_{\alpha\beta}$ are constants.[5]

Returning to the case of non-Abelian groups for which $q = r$, we observe from (1.23) that the quantities $L_{\beta\mu\nu}^\lambda$ are zero and from (1.29) that the system of equations (1.28) are completely integrable. Consequently it is possible to express the functions φ^λ in (1.24) as a power series by means of (1.28). In order to insure that the transformation be non-singular, the initial values of $\dfrac{\partial x^\lambda}{\partial x'^\nu}$ must be chosen so that their determinant is not equal to zero. The initial values of $\dfrac{\partial x^\lambda}{\partial x'^\sigma}$ may be taken arbitrarily, as also the derivatives of x^μ of any order with respect to $x^1, \ldots, x^q$, or with respect to $x^{q+1}, \ldots, x^n$. Hence we have the theorem:

When the rank of the matrix $\|\xi_a^\alpha\|$ *of an intransitive group* G_r *is* r, *there exists a coördinate system for the* V_n *in which the* ξ's *are functions of* $x^1, \ldots, x^r$ *at most, the minimum invariant varieties being defined by* $x^{r+1} = const., \ldots, x^n = const.$

3. *When* $q < r$.—Since by hypothesis the determinant $|g_{\alpha\beta}|$ is not zero, a set of functions $g^{\alpha\beta}$ is defined by

$$g^{\alpha\beta}g_{\beta\gamma} = \delta_\gamma^\alpha.$$

If we multiply (2.4) by $g^{\beta\sigma}$ for $\sigma = q+1, \ldots, n$ and sum for β, we have, because of (1.13),

$$\xi_h^\lambda g_{\gamma\lambda} g^{\beta\sigma}\frac{\partial \varphi_p^h}{\partial x^\beta} = 0.$$

If we multiply this equation by $g^{\gamma\mu}$ and sum for γ, and note that the determinant $|\xi_h^\lambda|$ is not zero, we have that the $q(n-q)$ functions φ_p^h are solutions of the $n-q$ equations

$$g^{\beta\sigma}\frac{\partial\varphi}{\partial x^{\beta}} = 0 \begin{pmatrix}\beta = 1, \ldots, n \\ \sigma = q+1, \ldots, n\end{pmatrix}. \tag{3.1}$$

Hence at most q of the φ's are independent.

Suppose that q of them are independent, and denote them by $\varphi^1, \ldots, \varphi^q$; then the matrix $\left\|\frac{\partial\varphi^h}{\partial x^\alpha}\right\|$, for $h = 1, \ldots, q;\ \alpha = 1, \ldots, n$, is of rank q. If the determinant $\left|\frac{\partial\varphi^h}{\partial x^\lambda}\right|$ for $\lambda = 1, \ldots, q$ is zero, there exist functions A_h such that

$$A_h\frac{\partial\varphi^h}{\partial x^\lambda} = 0. \tag{3.2}$$

Then from (3.1) we have

$$g^{\tau\sigma}A_h\frac{\partial\varphi^h}{\partial x^\tau} = 0 \qquad (\tau = q+1, \ldots, n) \tag{3.3}$$

Hence if $|g^{\tau\sigma}| \neq 0$, we have $A_h\frac{\partial\varphi^h}{\partial x^\tau} = 0$, which together with (3.2) imply that the φ's are not independent. But $|g^{\sigma\tau}| \neq 0$, if the fundamental quadratic form is definite.[4] Consequently the determinant $\left|\frac{\partial\varphi^h}{\partial x^\lambda}\right|$ is not zero in this case, and accordingly a non-singular transformation of coördinates is defined by

$$x'^{\lambda} = \varphi^{\lambda},\ x'^{\sigma} = x^{\sigma}. \tag{3.4}$$

In this new coördinate system (dropping primes) we have from the equations (1.16) for the functions φ^λ equations of the form

$$\Phi^\lambda_i\xi_{i\mu} = \delta^\lambda_\mu, \tag{3.5}$$

where the Φ^λ_i are certain of the functions Φ^h_{ip} and being functions of φ^λ are independent of $x^{q+1}, \ldots, x^n$. From these equations and (1.14) it follows that

$$\xi^\lambda_h = \Phi^\lambda_h. \tag{3.6}$$

From these equations and (1.7) it follows that ξ^λ_α are functions of φ^λ and consequently of $x^1, \ldots, x^q$ at most. Hence we have:

When q of the functions φ^h_p are independent and the fundamental quadratic form is definite, there exists a coördinate system for which (1.13) hold, and the components ξ^λ_a are functions of $x^1, \ldots, x^q$ at most.

From the preceding argument it follows also that:

When the fundamental form is indefinite and the coördinate system is such that (1.13) hold and there are q of the functions φ^h_p of which the jacobian with *respect to* $x^1, \ldots, x^q$ *is not zero, then in the coördinate system* (*3.4*) *the components* ξ'^{λ}_a *are functions of* $x^1, \ldots, x'^q$ *at most.*

We consider next the case when there are $s(< q)$ independent functions φ^h_p, denoted by $\varphi^1, \ldots, \varphi^s$. Suppose that the jacobian matrix of these φ's with respect to $x^1, \ldots, x^q$ is less than s. Then we have equations of the form (3.2) and (3.3) in which h takes the values 1 to s. Hence, unless the determinant $|g^{\sigma\tau}|$ is zero, we have from (3.3) that the rank of $\left\|\frac{\partial\varphi^h}{\partial x^\alpha}\right\|$ is less than s, contrary to hypothesis. If then the fundamental form is definite, there is no loss in generality in assuming that the determinant $\left|\frac{\partial\varphi^h}{\partial x^{\lambda\prime}}\right|$ for $\lambda_1, h = 1, \ldots, s$ is not zero. When we effect the transformation of coördinates

$$x'^{\lambda_1} = \varphi^{\lambda_1},\ x'^{\lambda_2} = x^{\lambda_2},\ x'^{\sigma} = x^{\sigma} \quad \begin{pmatrix} \lambda_1 = 1, \ldots, s; \\ \lambda_2 = s+1, \ldots, q; \\ \sigma = q+1, \ldots, n \end{pmatrix}$$

in the new coördinate system all the functions φ^h_p are functions of $x^1, \ldots, x^s$.

Proceeding as above, we have in place of (3.6) $\xi^{\lambda_1}_h = \Phi^{\lambda_1}_h$, where the Φ's are functions of $x^1, \ldots, x^s$. We apply now a transformation of coordinates

$$x'^{\lambda_1} = x^{\lambda_1}, \quad x'^{\lambda_2} = \varphi^{\lambda_2}(x'^1, \ldots, x'^n), \quad x'^{\sigma} = x^{\sigma},$$

after the manner of (1.24). In place of (1.28) we have

$$\frac{\partial^2 x^{\lambda_2}}{\partial x'^{\nu}\partial x'^{\sigma}} + L^{\lambda_2}_{\alpha\mu}\frac{\partial x^{\alpha}}{\partial x'^{\sigma}}\frac{\partial x^{\mu}}{\partial x'^{\nu}} = 0. \tag{3.7}$$

From (1.23) it follows that $L^{\lambda}_{\alpha\mu\nu} = 0$ for $\alpha \gtrless = s+1, \ldots, n$. Consequently equations (3.7) are completely integrable as follows from (1.29) for the system (3.7). Hence the first of the above theorems holds when the number of independent functions φ^h_p is less than q ,and we have the theorem:

When the fundamental quadratic form is definite there exists a coördinate system for which equations (*1.13*) *hold and the components* ξ^{λ}_a, *for* $\lambda = 1, \ldots, q$ *are functions of* $x^1, \ldots, x^q$ *at most.*

Also we have:

When the fundamental quadratic form is indefinite and there are $s(\gtrless q)$ *independent functions* φ^h_p, *if for a coördinate system in which* (1.13) *hold, the jacobian matrix of the s independent functions with respect to* $x^{\tau}, \ldots, x_q$ *if of rank s, there exists a coördinate system in which the components* ξ^{λ}_{α} *are functions of* $x^1, \ldots, x^q$ *at most.*

[1] "Sugli spazii che ammettono un gruppo continuo di movimenti," *Annali di Matematica*, ser. 3, **8**, p. 40 (1903).

[2] Cf. Eisenhart, *Riemannian Geometry*, p. 234, 1926.

[3] Cf. Fubini, loc. cit., p. 54.

[4] If the determinant $| g_{\lambda\mu} |$ for $\lambda, \mu = 1, \ldots, q, (q < n)$ is zero, real quantities ξ^λ are defined by the equations $g_{\lambda\mu}\xi^\lambda = 0$. The vector with the components ξ^λ, $\xi^\sigma = 0$ $(\sigma = q + 1, \ldots, n)$ is a real null vector, which is impossible, if the fundamental form is definite. In a similar manner it can be shown that the determinant $| g^{\sigma\tau} |$ for $\sigma, \tau = 1, \ldots, p$ cannot be zero for a definite form.

[5] Cf. Bianchi, *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*, p. 521, 1918.

ON CERTAIN PROPERTIES OF SEPARABLE SPACES

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1. Let R be a separable metric space, Δ a *normal development* of R, that is a sequence of finite coverings by open sets $\{\Sigma^i\} = \{\{U^{i\alpha}\}\}$, with the following properties: (*a*) the U's constitute a fundamental set of neighborhoods: (*b*) every $\overline{U}^{i+1}$ is on some U^i; (*c*) if $\overline{U}^{i\alpha} \subset U^{j\beta}$, for h high enough every $\overline{U}^{h\gamma}$ meeting $\overline{U}^{i\alpha}$ is on $U^{j\beta}$; (*d*) every decreasing sequence $\{U^{N_i\alpha_i}\}, (\overline{U}^{N_{i+1}\alpha_{i+1}} \subset U^{N_i\alpha_i})$ converges to a point or to zero. Developments of a very general type have been introduced by E. H. Moore, and many have been considered on repeated occasions in the theory of abstract spaces. In particular, spaces possessing a certain type called *regular* have been investigated with considerable success by Chittenden and Pitcher, notably in connection with the problem of metrization. Normal developments which are considered here for the first time appear to possess just the right degree of generality for the comfortable treatment of separable metric spaces. By means of them a number of the important theorems in dimension theory can be derived with little more trouble than in the compact case. Above all they have enabled us to obtain the proof as yet unavailable of a basic imbedding theorem on separable spaces.

2. The fundamental proposition is the

Existence theorem. R always possesses a normal development.

We shall only outline the proof here, reserving the details for publication elsewhere. Let us identify R with its topological image on the Hilbert parallelotope H image whose existence has been proved by Urysohn. Since H is compact it possesses a finite covering by open sets σ, and the intersections of its sets with R determine $\Sigma^1 = \{U^{1\alpha}\}$. We choose a metric such that there exists an $\eta > 0$ such that any subset of R whose diameter $< \eta$ is on a $U^{1\alpha}$. This is done by modifying the mapping on H, the new distance function always yielding distances greater than the old. In